Stochastic processes with stable distributions in random environments

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The asymptotic behavior in random environments of random flights with stable distribution laws is analyzed by the field-theoretic renormalization group. Random force fields with isotropic, divergenceless, curl-free, and unconstrained pair correlation functions with both finite and infinite correlation length are considered. Stability of the effective distribution laws in the scaling limit is determined and the scaling dimension of time is calculated in the $\varepsilon = d_c - d$ expansion, where d_c is the critical dimension of the model.

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I. INTRODUCTION

Random flights in random environments have attracted increasing attention (see, e.g., the recent review [1]), the emphasis having been in the analysis of the anomalous diffusion in these systems. In the case of ordinary Brownian motion the mean-square displacement of the position $\mathbf{r}$ of a test particle is proportional to the elapsed time. In a random environment the motion of the test particle is affected by a random force field, and anomalous diffusion, i.e., powerlike asymptotic behavior,

$$\langle \overline{r^2(t)} \rangle \propto t^{2\nu}, \tag{1}$$

where the scaling dimension of time $\nu \neq 1/2$ results. Here, the overbar denotes the average over the step distribution of the random flights, and the brackets the average over the distribution of the random force field.

The anomalous diffusive behavior of the random flights depends heavily on both the tensor structure and the long-distance asymptotic behavior of the force correlation function. In the simplest case of Brownian motion in isotropic random field with short-range correlations subdiffusive behavior with $\nu < 1/2$ is found [2] below the critical dimension $d_c = 2$, above which normal diffusion with $\nu = 1/2$ takes place.

When independent potential and solenoidal parts of the random field and powerlike long-range asymptotics of the correlations are allowed, the anomaly may be either subdiffusive or superdiffusive with $\nu > 1/2$ [3–5]. In general, the solenoidal part of the field gives rise to superdiffusive behavior, the potential part to subdiffusive behavior. The exponent of the assumed powerlike behavior of the long-range correlated random force shifts the value of the critical dimension, below which the anomaly occurs. Above and at the critical dimension the limiting distribution of the random flights is a stable distribution. Below the critical dimension, however, corrections to the stable law occur due to the force field fluctuations. A scaling form for the limiting distribution function $P(t,\mathbf{r})$ may be obtained from a renormalization-group argument $P(t,\mathbf{r}) = t^{-d\nu}Q(\mathbf{r}t^{-\nu})$, where the scaling function $Q(\mathbf{x})$ may be calculated in a $d_c - d$ expansion below the critical dimension.

Recently, the renormalization-group analysis of the limiting behavior of random flights was applied to the case of Lévy flights in an isotropic short-range correlated random field [6]. In the Lévy flights the step distribution falls off like a power of the step length $p(\eta) \propto \eta^{-1-\mu}$ with the step index $0 < \mu < 2$. From the Langevin equation for the position $\mathbf{r}$ of a test particle in the external field $\mathbf{F}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{F}(\mathbf{r}) + \boldsymbol{\eta}$$

it then follows that the distribution function $P(t,\mathbf{r}) = \overline{\delta(\mathbf{r} - \mathbf{r}(t))}$ obeys the following Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -D_1(-\nabla^2)^{\mu/2}P + D_2\nabla^2 P - \boldsymbol{\nabla}\cdot[\mathbf{F}P] \, . \tag{2}$$

Here, the fractional power of ∇^2 is defined through the Fourier transform. The ordinary diffusion term comes from the small-scale part of the step distribution. When $\mathbf{F} = \mathbf{0}$, the function

$$P(t,\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r} - t(D_1 k^\mu + D_2 k^2)}$$

yields the transition probability of stable stochastic processes [7]: $D_1 = 0$ corresponds to Brownian motion and $D_2 = 0$ to Lévy flights.

The zero-mean Gaussian distribution of the random field is determined by the correlation function

$$\langle F_i(\mathbf{r})F_j(\mathbf{r}')\rangle = C_{ij}(\mathbf{r} - \mathbf{r}') \, . \tag{3}$$

For the isotropic short-range correlated field the correlation function $C_{ij}(\mathbf{r} - \mathbf{r}') = g\delta_{ij}\delta(\mathbf{r} - \mathbf{r}')$, where the coupling constant g is a measure of the strength of disorder. A perturbative solution of the stochastic problem (2,3) becomes inconsistent below the critical dimension d_c, at and below which contributions from the small-momentum region in the Fourier-transformed problem give rise to effective coupling constants growing with time.

These infrared divergences may be dealt with by the use of the renormalization group at the critical di-

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mension, when they can be transferred to the large-momentum region. The results may be extended below the critical dimension in the form of a $d_c - d$ expansion [8]. The critical dimension in this case is $d_c = 2\mu - 2$, above which the perturbative solution is consistent and the effective large-scale solution for the Green function of the diffusion equation (2) in the random field is the same as the Green function in zero field, apart from a finite renormalization of the diffusion coefficient D_2. When $d < d_c$, the higher-order contributions to the Green function affect its structure and the resulting limiting distribution is not stable. The scaling dimension of time, however, turns out to be $\nu = 1/\mu$, independent of the space dimensionality [6]. In particular, it is not affected by the random field. This follows from that the short-range correlated random field gives rise to subdiffusive anomalous behavior, which does not compete with the superdiffusive behavior due to the Lévy flights. When the random field contains a dominating solenoidal part, however, superdiffusive behavior is expected due to both the Lévy flights and random force, and it is not obvious what the scaling dimension of time ν will be in this case.

Formally ν determines the long-time behavior of the moments of the displacement: $\langle \overline{r^n(t)} \rangle \propto t^{n\nu}$. However, the mean-square displacement (1) does not exist in the case of Lévy flights [9]. For Lévy flights with $1 < \mu < 2$ the first moment $\langle \overline{r(t)} \rangle \propto t^{\nu}$ is finite. Therefore the long-time distribution may be characterized by the behavior of the first moment instead of the mean-square displacement in this case.

In this work I analyze the interplay between the local and nonlocal terms in the diffusion equation (2) for various random fields treated in the literature for the case of Brownian motion. The problem of relative impact to the asymptotic behavior of local and nonlocal terms with almost equal scaling dimensions has occurred on several occasions in the past [5,10]. The interplay of the operators $(-\nabla^2)^{\mu/2}$ and ∇^2 in the ϕ^4 field theory has recently been analyzed [11], and I shall follow the approach of this work in the subsequent analysis of the random flight problem.

II. FIELD-THEORETIC RENORMALIZATION OF THE MODEL

I shall use the field-theoretic version of the renormalization group, since in this framework the analysis of the various asymptotic scaling regimes and the crossover between them may be carried out in a fairly general form. The Green function of the diffusion equation (2) averaged over the random force may be written in the form of a functional integral $G(t-t', \mathbf{r}-\mathbf{r}') = \int \mathcal{D}\varphi\mathcal{D}\tilde{\varphi}\mathcal{D}\mathbf{F}\, \varphi(t,\mathbf{r})\tilde{\varphi}(t',\mathbf{r}')e^{S}$ where the "action" is of the form

$$S = \int d\mathbf{r}dt\, \tilde{\varphi}\left[-\partial_t - D_1(-\nabla^2)^{\mu/2} + D_2\nabla^2\right]\varphi - \frac{1}{2}\int d\mathbf{r}d\mathbf{r}' \sum_{i,j} F_i(\mathbf{r})C^{-1}_{ij}(\mathbf{r}-\mathbf{r}')F_j(\mathbf{r}') + \int d\mathbf{r}dt\, \mathbf{F}\varphi\nabla\tilde{\varphi}\,. \qquad (4)$$

The standard renormalization theorem [8] is constructed for interactions local in space, therefore the number of variables may be reduced by integrating out the random force field when its correlation function is isotropic with short range. In this case the action may be cast in the form

$$S = \int d\mathbf{r}dt\, \tilde{\varphi}\left[-\partial_t - D_1(-\nabla^2)^{\mu/2} + D_2\nabla^2\right]\varphi + \frac{g}{2}\int d\mathbf{r}dtdt' \sum_i \varphi(t,\mathbf{r})\partial_i\tilde{\varphi}(t,\mathbf{r})\varphi(t',\mathbf{r})\partial_i\tilde{\varphi}(t',\mathbf{r})\,. \qquad (5)$$

In the generic case the correlation function [3–5] consists of independent transverse and longitudinal parts: $C = C^T + C^L$, where

$$C^T_{mn}(\mathbf{r}) = g_T\int \frac{d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{r}\cdot\mathbf{k}}}{k^{2\alpha}}\left(\delta_{mn} - \frac{k_mk_n}{k^2}\right)\,, \quad C^L_{mn}(\mathbf{r}) = g_L\int \frac{d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{r}\cdot\mathbf{k}}}{k^{2\alpha}}\frac{k_mk_n}{k^2}\,. \qquad (6)$$

The correlation function of the isotropic short-range case is recovered for $g_T = g_L$ and $\alpha = 0$. More sophisticated correlation functions have also been discussed [12]. However, the analysis is similar in all cases, therefore I do not dwell on them here.

For the analysis of the crossover regime the model is renormalized at $d = 2$ and $\mu = 2$. By the standard power counting [8] the action (5) is found to be multiplicatively renormalizable and thus the renormalized action is of the form

$$S = \int d\mathbf{r}dt\, \tilde{\varphi}\Big[-\partial_t - D_{1R}(-\nabla^2)^{\mu/2} + Z_2M^{-\delta}D_{2R}\nabla^2\Big]\varphi + \frac{g_R}{2}Z_1M^{\tilde{\varepsilon}}\int d\mathbf{r}dtdt'(\varphi\nabla\tilde{\varphi})(t,\mathbf{r})(\varphi\nabla\tilde{\varphi})(t',\mathbf{r})\,. \qquad (7)$$

Here, $\delta = 2-\mu$, $\tilde{\varepsilon} = 2\mu - 2 - d$, M is the scale-setting parameter and Z_1, Z_2 are the renormalization constants of the model, in which all ultraviolet divergences may be collected in a renormalizable model. It should be noted that the nonlocal operator $-D_1(-\nabla^2)^{\mu/2}$ is not renormalized even in the case when the renormalization is carried out for $\mu = 2$, when both diffusion terms are indistinguishable [13].

To facilitate the calculation of the renormalization constants, one of the diffusion terms is regarded as a part of the interaction. The choice of the scaling parameter M in the action (7) corresponds to that the term $\propto D_{2R}\nabla^2$ is treated as an interaction term in the calculation of the renormalization constants. When dimensional regularization is used and the parameter $\delta = 2-\mu$ is small, the effect of this interaction amounts to the shift of the location of the poles in $\tilde{\varepsilon}$ by the quantity $l\delta$, where l is the number of $\tilde{\varphi}\nabla^2\varphi$ insertions in the graph considered. The residues of the poles are the same as in the Brownian motion problem.

The connection between the renormalized and unrenormalized parameters is

$$Z_1 g_R M^{\tilde{\varepsilon}} = g\,,$$

$$Z_2 D_{2R} M^{-\delta} = D_2\,.$$

The renormalization-group functions are defined as $\gamma_i = -M\partial_M|_{D,g} \ln Z_i$, where the partial derivatives are calculated with fixed unrenormalized parameters. For the β functions

$$\beta_u = M\frac{\partial}{\partial M}\bigg|_{D,g} u_R\,,$$

$$\beta_\zeta = M\frac{\partial}{\partial M}\bigg|_{D,g} \zeta_R\,, \qquad (8)$$

which determine the asymptotic behavior of the effective coupling constants of the model, the following expressions are established in analogy with the ϕ^4 model [11]:

$$\beta_u = u_R\left[-\tilde{\varepsilon} + \gamma_{1D}(u_R) - 2\gamma_{2D}(u_R)\right] - \frac{2\delta u_R \zeta_R}{1+\zeta_R}\,,$$

$$\beta_\zeta = \zeta_R \delta + (1+\zeta_R)\gamma_{2D}(u_R)\,. \qquad (9)$$

Here, $u_R = g_R(D_1 + D_{2R})^{-2}$, $\zeta_R = D_{2R}/D_1$ are the dimensionless expansion parameters of the model. Remarkably enough, these β functions are completely expressed through the renormalization-group functions of the Brownian motion problem, which are labeled by the subscript D.

III. STABILITY OF THE ASYMPTOTIC SCALING REGIMES

Equations (8) and (9) for u_R and ζ_R are then linearized near the fixed points determined by the equations $\beta_u = \beta_\zeta = 0$. A fixed point governs the large-scale behavior of the model, when the matrix $\omega_{ab} = \partial\beta_a/\partial b$, where $(a,b)=(u_R,\zeta_R)$, has eigenvalues with positive real parts at the fixed point. The trivial fixed point $u_* = 0$, $\zeta_* = 0$ is infrared stable and determines the large-scale asymptotic behavior of the model, when the conditions

$$\tilde{\varepsilon} = \varepsilon - 2\delta < 0\,, \qquad \delta > 0 \qquad (10)$$

are satisfied. Here, $\varepsilon = 2 - d$. At this fixed point neither the ordinary diffusive term nor the random force affect the asymptotic behavior of the model. The limiting distribution is therefore the stable distribution of Lévy flights. This scaling regime shall be referred to as the ordinary Lévy flights. The nontrivial fixed point is found from the system of equations

$$\gamma_{1D}(u_*) = \tilde{\varepsilon}\,, \qquad \left[\delta + \gamma_{2D}(u_*)\right]\zeta_* = -\gamma_{2D}(u_*)\,, \quad (11)$$

which have a unique solution in the perturbation theory. At this fixed point the scaling dimension of time remains equal to $\nu = 1/\mu$, resulting in superdiffusive behavior. The scaling function Q, however, contains perturbative corrections and therefore the limiting distribution is not stable. The stability conditions of this fixed point are

$$\left[\delta + \gamma_{2D}(u_*)\right] u_* \gamma_{1D}'(u_*) > 0\,,$$

$$\delta + \gamma_{2D}(u_*) + u_*\left[\gamma_{1D}'(u_*) - 2\gamma_{2D}'(u_*)\right] > 0\,, \qquad (12)$$

where the prime denotes the derivative with respect to u.

For the correlation functions (6), the renormalization-group functions have been calculated to two-loop order [3,5]. For the isotropic short-range model they are $\gamma_{1D}(u) = u/2 + u^2/2 + O\left(u^3\right)$ and $\gamma_{2D}(u) = u^2/2 + O\left(u^3\right)$. From these expressions it can readily be seen that the conditions (12) are met to two-loop order in the ε expansion, provided the inequalities

$$\varepsilon - 2\delta > 0\,, \qquad \delta + \gamma_{2D}(u_*) > 0 \qquad (13)$$

are satisfied. It seems plausible that these conditions determine the region of stability of the nontrivial fixed point also in higher orders of the perturbation theory. At the leading order $\delta + \gamma_{2D}(u_*) = \delta + 2\varepsilon^2$, which means that the line $d < 2$, $\mu = 2$ lies in the region of stability of this scaling regime of anomalous Lévy flights. However, this line must be excluded from the region of stability, because the model of Brownian motion in random field [3] is recovered on it with corresponding scaling dimensions and functions. In particular, $\nu = [2 + \gamma_{2D}(u_*)]^{-1} = 1/2 - \varepsilon^2/2 + O(\varepsilon^3)$ leading to subdiffusive behavior. Thus, the scaling functions and dimensions of this model of random flights are discontinuous functions of d and μ. It should be noted that the Lévy flight term $-D_1(-\nabla^2)^{\mu/2}$ determines the asymptotic behavior of the model also for $\mu > 2$ in the region $\delta + \gamma_{2D}(u_*) > 0$, $\delta = 2 - \mu < 0$. Due to the random field, the asymptotic behavior does not reduce to ordinary diffusion when the step index $\mu > 2$, as it does in the free-field case.

For the investigation of the asymptotic behavior of the model for the values d and μ not covered by the conditions (10) and (13), the Lévy flight term is treated as a part of the interaction. This choice corresponds to the renormalized action of the form

$$S = \int d\mathbf{r}dt\,\tilde{\varphi}\left[-\partial_t - D_{1R}M^\delta(-\nabla^2)^{\mu/2} + Z_2 D_{2R}\nabla^2\right]\varphi$$
$$+\frac{g_R}{2} Z_1 M^\varepsilon \int d\mathbf{r}dtdt'(\varphi\nabla\tilde{\varphi})(t,\mathbf{r})(\varphi\nabla\tilde{\varphi})(t',\mathbf{r})\,.$$

Both diffusion coefficients are renormalized here: $D_2 = Z_2 D_{2R}$, $D_1 = M^\delta D_{1R}$. It is convenient to choose the ratio $D_{1R}/D_{2R} = \chi_R$ as a dimensionless expansion parameter. The β functions are

$$\beta_u = u_R\left[-\varepsilon + \gamma_{1D}(u_R) - 2\gamma_{2D}(u_R)\right]$$
$$+2\delta u_R \chi_R/(1+\chi_R)\,,$$
$$\beta_\chi = \chi_R[-\delta - (1+\chi_R)\gamma_{2D}(u_R)]\,. \qquad (14)$$

The trivial fixed point $u_* = 0$, $\chi_* = 0$, which corresponds to ordinary Brownian motion, is stable for $\delta < 0$ and $\varepsilon < 0$. The fixed point

$$\chi_* = 0\,, \qquad \gamma_{1D}(u_*) - 2\gamma_{2D}(u_*) = \varepsilon \qquad (15)$$

of the β functions (14), which corresponds to anomalous

TABLE I. Regions of stability in the (ε,δ) plane, and the scaling dimension ν for random flights in random fields with short-range (SR) or long-range (LR) correlations. The columns correspond to ordinary Brownian motion (OB), anomalous Brownian motion (AB), ordinary Lévy flights (OL), and anomalous Lévy flights (AL). The limit distribution is stable in the OB and OL regimes. The notation $\gamma_{2D}^{*}=\gamma_{2D}(u_{*})$ has been used for brevity.

Random field	OB $\nu=1/2$	AB $\nu=1/(2+\gamma_{2D}^{*})$	OL $\nu=1/\mu$	AL $\nu=1/\mu$
Isotropic SR	$\varepsilon<0,\ \delta<0$	$\delta+\gamma_{2D}^{*}<0$[a] or $\delta=0,\ \varepsilon>0$	$\varepsilon<2\delta,\ \delta>0$	$\delta+\gamma_{2D}^{*}>0$ $\delta\neq0,\ \varepsilon>2\delta$
.sotropic LR	$\varepsilon+2\alpha<0$ $\delta<0$	$\delta+\gamma_{2D}^{*}<0$[b] $\varepsilon+2\alpha>0$	$\varepsilon+2\alpha<2\delta$ $\delta>0$	$\delta+\gamma_{2D}^{*}>0$ $\varepsilon+2\alpha>2\delta$
Divergenceless SR	$\varepsilon<0,\ \delta<0$	$\varepsilon>0,\ \varepsilon>2\delta$[c]	$\varepsilon<2\delta,\ \delta>0$	
Divergenceless LR	$\varepsilon+2\alpha<0$ $\delta<0$	$\varepsilon+2\alpha>2\delta$[d] $\varepsilon+2\alpha>0$	$\varepsilon+2\alpha<2\delta$ $\delta>0$	
Curl-free SR	$\varepsilon<0,\ \delta<0$	$\varepsilon>0,\ \delta=0$[e]	$\varepsilon<2\delta,\ \delta>0$	$\varepsilon>2\delta$ $\delta\neq0,\ \varepsilon>0$
Curl-free LR	$\varepsilon+2\alpha<0$ $\delta<0$	$\varepsilon+2\alpha>0,\ \delta=0$[e]	$\varepsilon+2\alpha<2\delta$ $\delta>0$	$\varepsilon+2\alpha>2\delta$ $\delta\neq0,\ \varepsilon+2\alpha>0$
Unconstrained SR	$\varepsilon<0,\ \delta<0$	$\delta+\gamma_{2D}^{*}<0$[a] or $\delta=0,\ \varepsilon>0$	$\varepsilon<2\delta,\ \delta>0$	$\delta+\gamma_{2D}^{*}>0$ $\varepsilon>2\delta,\ \delta\neq0$
Unconstrained LR	$\varepsilon+2\alpha<0$ $\delta<0$	$\delta+\gamma_{2D}^{*}<0$[f] $\varepsilon+2\alpha>0$	$\varepsilon+2\alpha<2\delta$ $\delta>0$	$\delta+\gamma_{2D}^{*}>0$ $\varepsilon+2\alpha>2\delta$

[a]Here, $\gamma_{2D}^{*}=2\varepsilon^{2}+O(\varepsilon^{3})$.
[b]Here, $\gamma_{2D}^{*}=-\alpha(\varepsilon+2\alpha)/(1+2\alpha)+O((\varepsilon+2\alpha)^{2})$.
[c]Here, $\nu=2/(4-\varepsilon)$.
[d]Here, $\nu=2/(4-\varepsilon-2\alpha)$.
[e]Strong-coupling regime. The anomalous dimension cannot be calculated in perturbation theory.
[f]Here, $\gamma_{2D}^{*}=(\kappa-1-2\alpha)(\varepsilon+2\alpha)/2(1+2\alpha)+O((\varepsilon+2\alpha)^{2})$.

diffusion due to the random field [3], is stable when the inequalities

$$\delta+\gamma_{2D}(u_{*})<0\,,\qquad u_{*}[\gamma_{1D}'(u_{*})-2\gamma_{2D}'(u_{*})]>0$$

are satisfied. In the ε, δ expansion the latter condition implies $\varepsilon>0$. At this fixed point subdiffusive behavior takes place with $\nu=[2+\gamma_{2D}(u_{*})]^{-1}=1/2-\varepsilon^{2}/2+O(\varepsilon^{3})$. For the fixed point of anomalous Lévy flights $u_{*}\neq0$, $\chi_{*}\neq0$ the change of variable $\chi\to1/\zeta$ leads to the previous equations (11) and (12).

When the isotropic correlation function has an infinite correlation length corresponding to powerlike asympotic behavior $\propto k^{-2\alpha}$ at small wave numbers, the interaction in (5) is nonlocal in space and the action (4) with local nonlinear terms should be used. The renormalized action is

$$S=\int d\mathbf{r}dt\,\tilde\varphi\left[-\partial_t-D_1(-\nabla^2)^{\mu/2}+Z_2D_{2R}\nabla^2\right]\varphi$$
$$-\frac{M^{-(\tilde\varepsilon+2\alpha)}}{2g_R}\int d\mathbf{r}\mathbf{F}(-\nabla^2)^{\alpha}\mathbf{F}+Z_1^{1/2}\int d\mathbf{r}dt\,\mathbf{F}\varphi\nabla\tilde\varphi\,.$$

In this case the perturbative expression of γ_{2D} contains also the linear term, and this leads to drastic changes in the scaling behavior. For large enough values of α, the random force gives rise to superdiffusive behavior and the region of stability of the anomalous Lévy flights lies in the $\varepsilon>0$, $\delta>0$ quadrant of the (ε,δ) plane. Due to the competition of the two sources of anomalies, superdiffusive behavior with two different scaling dimensions is thus brought about: for anomalous Lévy flights $\nu=1/\mu$ as before, and for anomalous Brownian motion $\nu=1/2+(\alpha/4)(\varepsilon+2\alpha)/(1+2\alpha)+O[(\varepsilon+2\alpha)^{2}]>1/2$, too.

When the force field is divergenceless, the interaction term (4) is not renormalized [5] and $\gamma_1=0$. Due to this, there is no regime of anomalous Lévy flights. The borderline between ordinary Lévy flights and anomalous Brownian motion is given by $\varepsilon-2\delta=0$ in the short-range case and by $\varepsilon+2\alpha-2\delta=0$ in the long-range case. The scaling dimension of time for anomalous Brownian motion may be calculated perturbatively exactly [5] and $\nu=(2-\varepsilon/2)^{-1}$ for short-range and $\nu=[2-(\varepsilon+2\alpha)/2]^{-1}$ for long-rance correlations of random force. The anomalous behavior is thus superdiffusive.

Curl-free force field may be expressed through a potential function ψ: $\mathbf{F}=\nabla\psi$. The action (4) is multiplicatively renormalizable also in this case [14]. There is no anomalous Brownian motion, except for the case $\mu=2$, in which the β function is trivial: $\beta_u=-\varepsilon u_R$ and a strong-coupling regime sets in for $\varepsilon+2\alpha>0$.

In the correlation function (6) of unconstrained force field both coupling constants are present and the behavior of the system is quite different in the short-range and long-range cases. In the short-range case the asymptotic behavior is governed by the isotropic fixed points (11) and (15) for all $g_{LR}\neq0$ and $g_{TR}\neq0$. In the long-range case the ratio of longitudinal and transverse coupling constants $g_{LR}/g_{TR}=\kappa$ is invariant under renormalization. For the anomalous Brownian motion, the character of the anomaly depends on the ratio κ.

The regions of stability and scaling dimension ν for the various cases are summarized in Table I.

IV. CONCLUSION

In this work the interplay between Lévy flights and Brownian motion in transport processes in random velocity fields has been analyzed by the field-theoretic renormalization group. It is shown that the deviation from the ordinary diffusive behavior is brought about by both the fluctuations of the velocity field and the long-tail step distribution of the Lévy flights. The ultimate long-time asymptotic behavior may be both superdiffusive and subdiffusive depending on the characteristics of the random field and the Lévy flights. The limiting distribution of the random flights is stable in the asymptotic scaling regimes, in which the nonlinear term of the diffusion equation is asymptotically irrelevant. In other cases the limiting distribution has a more complex form for which an ε expansion may be constructed.

The scaling dimension of time ν and the regions of stability of the asymptotic regimes in the leading nontrivial order of the ε expansion are presented for isotropic, divergenceless, curl-free, and unconstrained random velocity fields with short-range or long-range correlations.

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